Quantum Circuit Exercise

Gate definitions

We use the computational basis $\{|0\rangle, |1\rangle\}$ for one-qubit Clifford gates and $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for two-qubit Clifford gates.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix},$$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The T gate is non-Clifford and acts as

$$T|0\rangle = |0\rangle, \qquad T|1\rangle = e^{i\pi/4}|1\rangle.$$

A note on Clifford gates and why the T gate matters

For students coming from quantum chemistry, a useful entry point to Clifford gates is through the Pauli operators. After a fermion-to-qubit transformation such as Jordan–Wigner, electronic-structure Hamiltonians are expressed as sums of Pauli strings. Clifford operations are special because they preserve this Pauli structure under conjugation:

$$\boxed{CPC^\dagger = P',}$$

where P and P' are Pauli strings (up to an overall phase). Equivalently, Clifford operations map the Pauli group onto itself.

The Clifford group is generated by the Hadamard H, phase S, and CNOT gates. The Hadamard, for example, exchanges the X and Z axes,

$$HXH = Z, \qquad HZH = X, \qquad HYH = -Y,$$

while the phase gate gives $SXS^\dagger = Y$ and leaves Z unchanged. CNOT performs the analogous transformation on two-qubit Pauli operators; for a control qubit 1 and target qubit 2,

$$X_1 \mapsto X_1X_2, \qquad X_2 \mapsto X_2, \qquad Z_1 \mapsto Z_1, \qquad Z_2 \mapsto Z_1Z_2.$$

Thus Clifford gates can create both superposition and entanglement. For example,

$$|00\rangle \xrightarrow{H_1} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\mathrm{CNOT}_{1\to 2}} \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

so a maximally entangled Bell state can be prepared using Clifford gates alone.

The remarkable point is that such circuits nevertheless possess an efficient classical description. A stabilizer state can be specified by the Pauli operators for which it is a $+1$ eigenstate. For example,

$$|00\rangle \leftrightarrow \{Z_1, Z_2\}, \qquad |\Phi^+\rangle \leftrightarrow \{X_1X_2, Z_1Z_2\}.$$

Because Clifford gates simply transform Pauli stabilizers into Pauli stabilizers, one need not explicitly propagate all 2^n wavefunction amplitudes. This is the content behind the Gottesman–Knill theorem: stabilizer circuits built from Clifford operations, suitable state preparation, and Pauli measurements can be simulated efficiently classically.

The T gate used in this exercise is different. It acts as

$$T|0\rangle = |0\rangle, \qquad T|1\rangle = e^{i\pi/4}|1\rangle,$$

and under conjugation

$$TXT^\dagger = \frac{X+Y}{\sqrt{2}},$$

which is no longer a single Pauli operator. Hence T is *non-Clifford*. Adding such a non-Clifford resource to Clifford gates makes universal quantum computation possible:

$$\boxed{\text{Clifford gates} + T \quad \Longrightarrow \quad \text{universal gate set.}}$$

For quantum chemistry, the useful conceptual message is that a generic correlated electronic wavefunction is not a stabilizer state. Clifford/stabilizer structure identifies a particularly classically tractable subset of Hilbert space, whereas non-Clifford resources allow us to leave that subset. One should not identify "non-Cliffordness" directly with electron correlation—they are distinct concepts—but it is meaningful to ask how much non-Clifford resource is needed to represent or prepare a correlated electronic state.

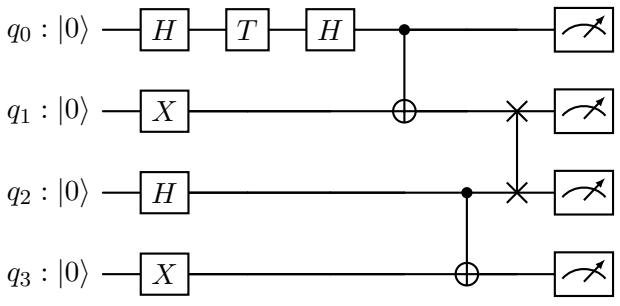
Connection to this exercise. The circuit deliberately contains both types of operations. The H, X, CNOT, and SWAP gates are Clifford operations, whereas T is non-Clifford. On q_0, the sequence $H \to T \to H$ makes the distinction visible: the first Hadamard creates a superposition, the T gate introduces a relative phase $e^{i\pi/4}$, and the second Hadamard converts that phase information through interference into unequal measurement probabilities. The two CNOTs then illustrate that Clifford gates can create entanglement and redistribute computational-basis amplitudes without themselves supplying the non-Clifford resource.

Exercise

The four qubits start in

$$|\psi_0\rangle = |0000\rangle,$$

where basis states are written as $|q_0q_1q_2q_3\rangle$. Consider the circuit



Here the CNOTs are $q_0 \to q_1$ and $q_2 \to q_3$, and the final SWAP exchanges q_1 and q_2.

1. Write down the state after the initial X and H gates.
2. Determine $HTH\,|0\rangle$ explicitly.
3. Write the full four-qubit state after the two CNOT gates.
4. Apply the SWAP $q_1 \leftrightarrow q_2$.
5. Which computational-basis states can be observed at the end?
6. Calculate the exact probability of measuring each of the 16 possible bit strings.
7. Verify that the probabilities sum to one.