# Quantum Circuit Exercise

## Solution

### 1. Single-qubit gates

For $q_0$ we first evaluate the non-Clifford sequence $HTH$:

$$\begin{aligned} H|0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \\ TH|0\rangle &= \frac{|0\rangle + e^{i\pi/4}|1\rangle}{\sqrt{2}}, \\ HTH|0\rangle &= \frac{1}{2}\Big[(1+e^{i\pi/4})|0\rangle + (1-e^{i\pi/4})|1\rangle\Big]. \end{aligned}$$

Define

$$a = \frac{1+e^{i\pi/4}}{2}, \qquad b = \frac{1-e^{i\pi/4}}{2}.$$

Thus $q_0$ is in the state $a|0\rangle + b|1\rangle$. Their squared magnitudes are

$$|a|^2 = \cos^2\frac{\pi}{8} = \frac{2+\sqrt{2}}{4}, \qquad |b|^2 = \sin^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{4}.$$

The remaining single-qubit gates give

$$q_1 : |0\rangle \xrightarrow{X} |1\rangle, \qquad q_2 : |0\rangle \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad q_3 : |0\rangle \xrightarrow{X} |1\rangle.$$

Therefore, just before the CNOT gates,

$$|\psi\rangle = (a|0\rangle + b|1\rangle)_{q_0} \otimes |1\rangle_{q_1} \otimes \left.\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right._{q_2} \otimes |1\rangle_{q_3}.$$

Expanding the tensor product gives

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left[a|0101\rangle + a|0111\rangle + b|1101\rangle + b|1111\rangle\right].}$$

### 2. First CNOT: $q_0 \to q_1$

For a CNOT with $q_0$ as control and $q_1$ as target,

$$|q_0q_1q_2q_3\rangle \longrightarrow |q_0,\, q_1 \oplus q_0,\, q_2,\, q_3\rangle.$$

Hence $q_1$ is flipped only if the control qubit $q_0$ is 1. Applying this rule term by term,

$$\begin{aligned} |0101\rangle &\longrightarrow |0101\rangle, \\ |0111\rangle &\longrightarrow |0111\rangle, \\ |1101\rangle &\longrightarrow |1001\rangle, \\ |1111\rangle &\longrightarrow |1011\rangle. \end{aligned}$$

The state after the first CNOT is therefore

$$\boxed{|\psi'\rangle = \frac{1}{\sqrt{2}}\left[a\,|0101\rangle + a\,|0111\rangle + b\,|1001\rangle + b\,|1011\rangle\right].}$$

Equivalently, focusing only on $q_0 q_1$,

$$(a\,|0\rangle + b\,|1\rangle)\,|1\rangle \xrightarrow{\text{CNOT}} a\,|01\rangle + b\,|10\rangle\,.$$

Thus the CNOT correlates $q_0$ and $q_1$ and, for nonzero $a$ and $b$, produces an entangled two-qubit state.

## 3. Second CNOT: $q_2 \to q_3$

For the second CNOT,

$$|q_0 q_1 q_2 q_3\rangle \longrightarrow |q_0,\, q_1,\, q_2,\, q_3 \oplus q_2\rangle\,.$$

Now $q_3$ is flipped only when $q_2 = 1$:

$$\begin{aligned}
|0101\rangle &\longrightarrow |0101\rangle\,,\\
|0111\rangle &\longrightarrow |0110\rangle\,,\\
|1001\rangle &\longrightarrow |1001\rangle\,,\\
|1011\rangle &\longrightarrow |1010\rangle\,.
\end{aligned}$$

Therefore

$$\boxed{|\psi''\rangle = \frac{1}{\sqrt{2}}\left[a\,|0101\rangle + a\,|0110\rangle + b\,|1001\rangle + b\,|1010\rangle\right].}$$

Again, looking only at $q_2 q_3$,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}\,|1\rangle \xrightarrow{\text{CNOT}} \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

which is a Bell state.

## 4. SWAP $q_1 \leftrightarrow q_2$

The SWAP acts as

$$|q_0 q_1 q_2 q_3\rangle \longrightarrow |q_0 q_2 q_1 q_3\rangle\,.$$

Thus

$$\begin{aligned}
|0101\rangle &\longrightarrow |0011\rangle\,,\\
|0110\rangle &\longrightarrow |0110\rangle\,,\\
|1001\rangle &\longrightarrow |1001\rangle\,,\\
|1010\rangle &\longrightarrow |1100\rangle\,.
\end{aligned}$$

The final state is

$$\boxed{|\psi_{\text{final}}\rangle = \frac{1}{\sqrt{2}}\left[a\,|0011\rangle + a\,|0110\rangle + b\,|1001\rangle + b\,|1100\rangle\right].}$$

## 5. Complete basis-state bookkeeping

The action of the two CNOTs and the final SWAP can be summarized compactly as

| Before CNOTs | $q_0 \to q_1$ | $q_2 \to q_3$ | SWAP $(q_1, q_2)$ |
|---|---|---|---|
| 0101 | 0101 | 0101 | 0011 |
| 0111 | 0111 | 0110 | 0110 |
| 1101 | 1001 | 1001 | 1001 |
| 1111 | 1011 | 1010 | 1100 |

The gates only permute these computational-basis labels; they do not change the coefficients attached to the corresponding terms. The unequal final probabilities originate from the earlier $HTH$ sequence on $q_0$.

## 6. Measurement probabilities

The amplitude of each state containing $a$ is $a/\sqrt{2}$, while the amplitude of each state containing $b$ is $b/\sqrt{2}$. Hence

$$\frac{|a|^2}{2} = \frac{2+\sqrt{2}}{8}, \qquad \frac{|b|^2}{2} = \frac{2-\sqrt{2}}{8}.$$

Only four computational-basis states have nonzero probability:

| Measurement | Probability |
|---|---|
| 0011 | $\frac{2+\sqrt{2}}{8} \approx 0.4268$ |
| 0110 | $\frac{2+\sqrt{2}}{8} \approx 0.4268$ |
| 1001 | $\frac{2-\sqrt{2}}{8} \approx 0.0732$ |
| 1100 | $\frac{2-\sqrt{2}}{8} \approx 0.0732$ |
| all other bit strings | 0 |

Finally,

$$2\frac{2+\sqrt{2}}{8} + 2\frac{2-\sqrt{2}}{8} = 1,$$

as required.

**Teaching point.** The $T$ gate initially introduces a relative phase. The second Hadamard converts this phase information into unequal populations, illustrating the chain

$$\boxed{\text{relative phase} \longrightarrow \text{interference} \longrightarrow \text{measurement probability.}}$$

The CNOT and SWAP gates then provide a clean exercise in tracking how amplitudes are reassigned to different computational-basis states.